



Note

On choosability of some complete multipartite graphs and Ohba's conjecture[☆]

Yufa Shen^{a,*}, Wenjie He^b, Guoping Zheng^a, Yanning Wang^c, Lingmin Zhang^a

^aDepartment of Mathematics and Physics, Hebei Normal University of Science and Technology, Qinhuangdao 066004, PR China

^bApplied Mathematics Institute, Hebei University of Technology, Tianjin 300130, PR China

^cSchool of Science, Yanshan University, Qinhuangdao 066004, PR China

Received 22 April 2005; received in revised form 16 March 2007; accepted 28 March 2007

Available online 3 April 2007

Abstract

A graph G is said to be chromatic-choosable if $\text{ch}(G) = \chi(G)$. Ohba has conjectured that every graph G with $2\chi(G) + 1$ or fewer vertices is chromatic-choosable. It is clear that Ohba's conjecture is true if and only if it is true for complete multipartite graphs. But for complete multipartite graphs, the graphs for which Ohba's conjecture has been verified are nothing more than $K_{3*2, 2*(k-3), 1}$, $K_{3, 2*(k-1)}$, and $K_{s+3, 2*(k-s-1), 1*s}$. These results have been obtained indirectly from the investigation about complete multipartite graphs by Gravier and Maffray and by Enomoto et al. In this paper we show that Ohba's conjecture is true for complete multipartite graphs $K_{4, 3, 2*(k-4), 1*2}$ and $K_{5, 3, 2*(k-5), 1*3}$. By the way, we give some discussions about a result of Enomoto et al.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C15

Keywords: List coloring; Complete multipartite graphs; Chromatic-choosable graphs; Ohba's conjecture

1. Introduction

List colorings are generalizations of usual colorings that were introduced independently by Vizing [11], and by Erdős et al. [3]. For a graph G and each vertex $u \in V(G)$, let $L(u)$ denote a list of colors available for u , $L = \{L(u) | u \in V(G)\}$ is said to be a list assignment of G . If $|L(u)| = k$ for all $u \in V(G)$, L is called a k -list assignment of G . An L -coloring from a given list assignment L is a proper coloring c , i.e., $c(u) \neq c(v)$ whenever $uv \in E(G)$, for every $u, v \in V(G)$, satisfying that $c(u) \in L(u)$ for every $u \in V(G)$. We call a graph G to be L -colorable if G admits an L -coloring. A graph G is called k -choosable if G is L -colorable for every k -list assignment L . The choice number $\text{ch}(G)$ of a graph G is the smallest k such that G is k -choosable.

Clearly, $\text{ch}(G) \geq \chi(G)$ holds for every graph G , where $\chi(G)$ denotes the chromatic number of G . On the other hand, Erdős et al. showed that bipartite graphs can have arbitrarily large choice number. It is significant to investigate the

[☆] Research supported in part by the Doctoral Foundation of Hebei Normal University of Science and Technology, PR China, and the Science Foundation of the Education Department of Hebei Province, PR China (Grant No. 2005108).

* Corresponding author.

E-mail address: syf030514@163.com (Y. Shen).

conditions or give some graph classes, in which each graph satisfies $\text{ch}(G) = \chi(G)$. For convenience, a graph G is called *chromatic-choosable*, if $\text{ch}(G) = \chi(G)$ [8]. About the chromatic-choosable graphs, some results and conjectures have been obtained, such as the famous *list chromatic conjecture* (see [6]), and the positive answer for line graphs of bipartite graphs (see [4]) (for more information we refer the interested reader to Alon [1] and Woodall [12]). Here we focus our attention on Ohba's conjecture:

Conjecture 1.1 (Ohba [8]). If $|V(G)| \leq 2\chi(G) + 1$, then $\text{ch}(G) = \chi(G)$.

For Conjecture 1.1, some special cases have been verified from the results of choice number of some complete multipartite graphs. We use the notation K_{l*r} for a complete r -partite graph in which each part is of size l . Notations such as $K_{l*r, m*t}$, etc. are used similarly. With the above notation, we restate the results of choice number of some complete multipartite graphs as follows:

Theorem 1.1 (Erdős et al. [3]). $\text{ch}(K_{2*k}) = k$.

Theorem 1.2 (Kierstead [7]). $\text{ch}(K_{3*k}) = \lceil (4k - 1)/3 \rceil$.

Theorem 1.3 (Gravier and Maffray [5]). If $k \geq 3$, $\text{ch}(K_{3*2, 2*(k-2)}) = k$.

Theorem 1.4 (Enotoma et al. [2]). $\text{ch}(K_{4, 2*(k-1)}) = \begin{cases} k & \text{if } k \text{ is odd;} \\ k + 1 & \text{if } k \text{ is even.} \end{cases}$

Theorem 1.5 (Enotoma et al. [2]). If $m \leq 2s + 1$, $\text{ch}(K_{m, 2*(k-s-1), 1*s}) = k$.

Theorem 1.6 (Ohba [9]). $\text{ch}(K_{3*r, 1*t}) = \max(r + t, \lceil (4r + 2t - 1)/3 \rceil)$.

From Theorems 1.3–1.5, it is clear that Ohba's conjecture is true for $K_{3*2, 2*(k-3), 1}$, $K_{3, 2*(k-1)}$, $K_{4, 2*(k-2), 1}$ and $K_{s+3, 2*(k-s-1), 1*s}$, and all k -partite subgraphs of them. By Theorem 1.6, let $r = t + 1$ and $k = r + t$, we know that $\text{ch}(K_{3*(t+1), 1*t}) = k$. Namely, Conjecture 1.1 is true for $K_{3*(t+1), 1*t}$ and its all k -partite subgraphs.

As a general situation, Reed and Sudakov [10] gave a weaker version of Ohba's conjecture. They showed that

Theorem 1.7 (Reed and Sudakov [10]). $\text{ch}(G) = \chi(G)$ provided $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$.

Furthermore, for the graphs with independence number at most three, as a weaker version of Ohba's conjecture, Ohba [9] proved that

Theorem 1.8 (Ohba [9]). Let G be a graph with $|V(G)| \leq 2\chi(G)$. If the independence number of G is at most 3, then G is chromatic-choosable.

Because every χ -chromatic graph is a subgraph of a complete χ -partite graph, Ohba's conjecture is true if and only if it is true for complete χ -partite graph. Namely, Conjecture 1.1 is equivalent to Conjecture 1.2.

Conjecture 1.2. If G is a complete k -partite graph with $|V(G)| = 2k + 1$, then $\text{ch}(G) = \chi(G) = k$.

In this paper, we will show that Conjecture 1.2 is true for another two special graph classes. In Section 2, we study some questions about a k -list assignment L of G , where $|V(G)| = 2k + 1$. We introduce or establish some lemmas and propositions involving the conditions on the list assignment L which ensure that such a graph G is L -colorable or G is not L -colorable. Using these lemmas and propositions, we will show that $\text{ch}(K_{4, 3, 2*(k-4), 1*2}) = k$, $\text{ch}(K_{5, 3, 2*(k-5), 1*3}) = k$ in Sections 3 and 4, respectively. Namely, for complete multipartite graphs $K_{4, 3, 2*(k-4), 1*2}$ and $K_{5, 3, 2*(k-5), 1*3}$, we show that Ohba's conjecture is true. In Section 5, we give some discussions about Theorem 1.5. The techniques of our proof in this paper are mainly from Refs. [7, 2].

2. Some lemmas and propositions

For a graph G and a subset $X \subset V(G)$, let $G[X]$ denote the subgraph of G induced by X . For a list assignment L of G , let $L|_X$ denote L restricted to X , and $L(X)$ denote the union $\bigcup_{u \in X} L(u)$. If A is a set of colors, let $L \setminus A$ denote the list assignment obtained from L by removing the colors in A from each $L(u)$ with $u \in V(G)$. When A consists of a single color a , we write $L - a$ instead of $L \setminus \{a\}$.

We say that L satisfies *Hall's condition* in G , if $|L(X)| \geq |X|$ for every subset $X \subset V(G)$. It is clear that if L satisfies Hall's condition, then by Hall's marriage theorem, there exists an L -coloring in which all vertices receive distinct colors.

In [7], the following lemma is proved. Here the statement is slightly different.

Lemma 2.1 (Kierstead [7]). *Let L be a list assignment for a graph G . Then G is L -colorable if $G[X]$ is $L|_X$ -colorable for a maximal non-empty subset $X \subset V(G)$ such that $|L(X)| < |X|$.*

Lemma 2.2. *Let $G = K_{3,1,1}$ with three parts $Y = \{y_1, y_2, y_3\}$, $W_1 = \{w_1\}$, $W_2 = \{w_2\}$, and L be a list assignment on the vertices of G with $|L(y_1)| = |L(y_2)| = 2$, $|L(y_3)| = 3$, $|L(w_1)| = |L(w_2)| = 2$, and $L(w_1) \neq L(w_2)$. Then G is L -colorable.*

Proof. *Case 1:* $L(y_1) \cap L(y_2) \neq \emptyset$. Let $a \in L(y_1) \cap L(y_2)$. Since $|L(w_1)| = |L(w_2)| = 2$ and $L(w_1) \neq L(w_2)$, we can choose $a_1 \in L(w_1)$ and $a_2 \in L(w_2)$ such that a, a_1 and a_2 are pairwise different. Let $c(y_1) = c(y_2) = a$, $c(w_1) = a_1$, $c(w_2) = a_2$, $c(y_3) \in L(y_3) \setminus \{a_1, a_2\}$, then c is an L -coloring for G .

Case 2: $L(y_1) \cap L(y_2) = \emptyset$. Consider the graph $G' = G - y_3$, it is easy to see that $L|_{V(G')}$ satisfies Hall's condition in G' , so G' admits an $L|_{V(G')}$ -coloring c' . Let $c(y_i) = c'(y_i)$ for $i = 1, 2$, $c(w_i) = c'(w_i)$ for $i = 1, 2$, $c(y_3) \in L(y_3) \setminus \{c(w_1), c(w_2)\}$, then c is an L -coloring for G . \square

Let $G = K_{m_1, m_2, 2*r, 1*s}$ be any complete k -partite graph with $|V(G)| = 2k + 1$, where $m_1 \geq m_2 \geq 3$, $r \geq 0$, $s \geq 1$, $2 + r + s = k$ and $m_1 + m_2 + 2r + s = 2k + 1$. Denote the k parts of G as $V_1 = \{x_1, x_2, \dots, x_{m_1}\}$, $V_2 = \{y_1, y_2, \dots, y_{m_2}\}$, $U_i = \{u_i, v_i\}$ for $i = 1, 2, \dots, r$, and $W_i = \{w_i\}$ for $i = 1, 2, \dots, s$. Suppose that L is a k -list assignment of G such that G is not L -colorable. Under the above assumption, we have the following propositions.

Proposition 2.1. $\bigcap_{x_i \in V_1} L(x_i) = \emptyset$, $\bigcap_{y_i \in V_2} L(y_i) = \emptyset$.

Proof. Suppose that there exists a color $a \in \bigcap_{x_i \in V_1} L(x_i)$. Then, assign a to all vertices x_i for $i = 1, 2, \dots, m_1$. Note that $G' = G - V_1 = K_{m_2, 2*r, 1*s}$ with $m_2 = 2k + 1 - m_1 - 2r - s = 2(2 + r + s) + 1 - m_1 - 2r - s = s + 5 - m_1 \leq s + 2 \leq 2s + 1$, and $L' = L - a$ with $|L'(u)| \geq k - 1$ for all $u \in V(G')$. By Theorem 1.5, G' is $(k - 1)$ -choosable. Hence we can obtain an L -coloring of G , a contradiction. Similarly, $\bigcap_{y_i \in V_2} L(y_i) = \emptyset$. \square

Proposition 2.2. *If $r = 0$ or $L(u_i) \cap L(v_i) = \emptyset$ for $i = 1, 2, \dots, r$, $r \neq 0$, then there exists $x_{i_1}, x_{i_2} \in V_1$ and $y_{i_1}, y_{i_2} \in V_2$ such that $L(x_{i_1}) \cap L(x_{i_2}) \neq \emptyset$ and $L(y_{i_1}) \cap L(y_{i_2}) \neq \emptyset$.*

Proof. Without loss of generality, suppose that $L(x_1), L(x_2), \dots, L(x_{m_1})$ are pairwise disjoint, then there must exist two vertices $y_{i_1}, y_{i_2} \in V_2$ such that $L(y_{i_1}) \cap L(y_{i_2}) \neq \emptyset$. Otherwise, it is easy to see that L satisfies Hall's condition. This contradicts to that G is not L -colorable. Let A be a largest subset of V_2 such that $\bigcap_{y \in A} L(y) \neq \emptyset$. By Proposition 2.1, we know that $2 \leq |A| \leq m_2 - 1$. Choose a color $a \in \bigcap_{y \in A} L(y)$, and let $G' = G - A$, $L' = L - a$. Then $|L'(x_i) \cup L'(x_j)| \geq 2k - 1$ for every $i, j = 1, 2, \dots, m_1$, $i \neq j$, $|L'(y_i)| = k$ for every $y_i \in V_2 \setminus A$, and $|L'(u_i) \cup L'(v_i)| \geq 2k - 1$ for $i = 1, 2, \dots, r$. Since G is not L -colorable, G' is not L' -colorable. In particular, L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. Clearly, $|X \cap V_1| \leq 1$ and $|X \cap U_i| \leq 1$ for $i = 1, 2, \dots, r$. Otherwise $2k - 1 \leq |L'(X)| < |X| \leq |V(G')| \leq 2k - 1$. This is a contradiction. Hence $|X \setminus V_2| \leq k - 1$. Note that $|L'(u)| \geq k - 1$ for every $u \in X \setminus V_2$ and $|L'(u)| = k$ for every $u \in X \cap V_2$. It is clear that $G'[X]$ is $L'|_X$ -colorable. By Lemma 2.1, G' is L' -colorable. This is a contradiction. \square

3. Ohba's conjecture is true for graphs $K_{4,3,2*(k-4),1*2}(k \geq 4)$

In order to prove that $\text{ch}(K_{4,3,2*(k-4),1*2}) = k$ by induction, we shall show that $\text{ch}(K_{4,3,1,1}) = 4$ first.

Theorem 3.1. $\text{ch}(K_{4,3,1,1}) = 4$.

Proof. For $G = K_{4,3,1,1}$, denote its four parts as $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{y_1, y_2, y_3\}$, $W_i = \{w_i\}$ for $i = 1, 2$. By contradiction, assume that L is a list assignment with $|L(u)| = 4$ for each $u \in G$ such that G is not L -colorable.

Let A be a largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \emptyset$, then we know that $2 \leq |A| \leq 3$ by Propositions 2.1 and 2.2. Choose a color $c_1 \in \bigcap_{x \in A} L(x)$ to color the vertices in A . Let $G' = G - A$, $L' = L - c_1$. As G is not L -colorable, G' is not L' -colorable. In particular, L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. In the following, we will prove that $G'[X]$ is $L'|_X$ -colorable. Then G' is L' -colorable by Lemma 2.1. Thus, we obtain a contradiction.

By the maximality of A , we have that $|L'(x)| = 4$ for every $x \in V_1 \setminus A$. And by Proposition 2.1, we assume, without loss of generality, that $|L'(y_1)| \geq 3$, $|L'(y_2)| \geq 3$ and $|L'(y_3)| = 4$. We also know that $|L'(w_i)| \geq 3$ for $i = 1, 2$.

Note that $|X \cap V_2|$ is at least one (as $|L'(X)| < |X|$). We consider three cases according to the size of $X \cap V_2$.

Case 1: $|X \cap V_2| = 1$.

In this case, $|X \setminus V_1| \leq 3$. As $|L'(u)| \geq 3$ for every $u \in X \setminus V_1$, and $|L'(x)| = 4$ for every $x \in X \cap V_1$, it is easy to see that $G'[X]$ is $L'|_X$ -colorable.

Case 2: $|X \cap V_2| = 2$.

Denote by y_p and y_q the two vertices of $X \cap V_2$, and by y_t the remaining vertex of V_2 . Clearly, $X \subseteq \{y_p, y_q, w_1, w_2\} \cup (V_1 \setminus A)$, so $|X| \leq 6$.

If $L'(y_p) \cap L'(y_q) = \emptyset$, we have $6 \leq |L(X)| < |X| \leq 6$. This is a contradiction.

If $L'(y_p) \cap L'(y_q) \neq \emptyset$, choose a color $b \in L'(y_p) \cap L'(y_q)$. Note that $|L'(x)| = 4$ for every $x \in V_1 \setminus A$, $|L'(y_i)| \geq 3$ for $i = 1, 2, 3$, and $|L'(w_i)| \geq 3$ for $i = 1, 2$. Let $c(y_p) = c(y_q) = b$, $c(w_1) \in L'(w_1) - b$, $c(w_2) \in L'(w_2) - b - c(w_1)$, $c(x) \in L'(x) - b - c(w_1) - c(w_2)$ for every $x \in V_1 \setminus A$. Then c is an $L'|_X$ -coloring of $G'[X]$.

Case 3: $|X \cap V_2| = 3$.

In this case, $\{y_1, y_2, y_3\} \subset X$.

Claim 3.1. $|L'(y_1) \cup L'(y_2) \cup L'(y_3)| \leq 6$.

Otherwise, $7 \leq |L'(X)| < |X| \leq |V(G')| \leq 7$. This is a contradiction.

Claim 3.2. $|A| = 2$, $|X \cap V_1| = 2$ and $|\bigcap_{x \in X \cap V_1} L'(x)| \geq 2$.

Suppose that the claim is not true, then $|A| = 3$, or $|A| = 2$ and $|X \cap V_1| \leq 1$, or $|A| = 2$, $|X \cap V_1| = 2$ and $|\bigcap_{x \in X \cap V_1} L'(x)| < 2$.

If $|A| = 3$, or $|A| = 2$ and $|X \cap V_1| \leq 1$, view the above two situations as a whole, we have $|X \cap V_1| \leq 1$ and $|X| \leq 6$. We only need to consider $L'(y_1)$ and $L'(y_2)$. If $L'(y_1) \cap L'(y_2) = \emptyset$, then $6 \leq |L'(X)| < |X| \leq 6$. This is a contradiction. If $L'(y_1) \cap L'(y_2) \neq \emptyset$, denote by b a color of $L'(y_1) \cap L'(y_2)$. Note that $|L'(x)| = 4$ for every $x \in V_1 \setminus A$, $|L'(y_1)| \geq 3$, $|L'(y_2)| \geq 3$ and $|L'(y_3)| = 4$, $|L'(w_i)| \geq 3$ for $i = 1, 2$. Let $c(y_1) = c(y_2) = b$, $c(w_1) \in L'(w_1) - b$, $c(w_2) \in L'(w_2) - b - c(w_1)$. If there exists a vertex $x \in X \cap V_1$, let $c(x) \in L'(x) - b - c(w_1) - c(w_2)$, $c(y_3) \in L'(y_3) - c(w_1) - c(w_2) - c(x)$. If $X \cap V_1 = \emptyset$, let $c(y_3) \in L'(y_3) - c(w_1) - c(w_2)$. Then c is an $L'|_X$ -coloring of $G'[X]$.

If $|A| = 2$, $|X \cap V_1| = 2$ and $|\bigcap_{x \in X \cap V_1} L'(x)| < 2$, then $|\bigcup_{x \in X \cap V_1} L'(x)| \geq 7$. Thus, we have that $7 \leq |L'(X)| < |X| \leq |V(G')| \leq 7$. This is a contradiction.

Since $|A| = 2$, $|X \cap V_1| = 2$ and $|\bigcap_{x \in X \cap V_1} L'(x)| \geq 2$, without loss of generality, let $A = \{x_3, x_4\}$, $X \cap V_1 = V_1 \setminus A = \{x_1, x_2\}$, and let $\{c_2, c'_2\} \subset L(x_1) \cap L(x_2)$ (note that $L'(x) = L(x)$ for every $x \in V_1 \setminus A$). Clearly, $|L(x_3) \cap L(x_4)| \geq 2$. The reason is that we can replace A by $\{x_1, x_2\}$, and c_1 by c_2 , and can obtain an assertion similar to Claim 3.2. Let $\{c_1, c'_1\} \subset L(x_3) \cap L(x_4)$.

Claim 3.3. $L(w_1) = L(w_2)$.

Suppose that $L(w_1) \neq L(w_2)$. Let $c(x_1) = c(x_2) = c_2$, and $G'' = G' - \{x_1, x_2\}$, $L'' = L' - c_2$. Then we have that $|L''(y_i)| \geq 2$ for $i = 1, 2$, and $|L''(y_3)| \geq 3$ by Proposition 2.1. We also know that $|L''(w_i)| \geq 2$ for $i = 1, 2$. Since $L(w_1) \neq L(w_2)$, we can always remove some colors from $L''(y_i)$ for $i = 1, 2, 3$, and $L''(w_i)$ for $i = 1, 2$, and obtain a

new list assignment L''' of G'' which satisfies $|L'''(y_1)| = |L'''(y_2)| = 2$, $|L'''(y_3)| = 3$, $|L'''(w_1)| = |L'''(w_2)| = 2$ and $L'''(w_1) \neq L'''(w_2)$. By Lemma 2.2, G'' is L''' -colorable, and hence L'' -colorable. Thus, $G'[X]$ is $L'|_X$ -colorable.

Claim 3.4. $L(w_1) = L(w_2) = \{c_1, c'_1, c_2, c'_2\}$.

Suppose that the claim is not true. If $c_1 \notin L(w_1) = L(w_2)$, similarly to the proof of Claim 3.3, we can obtain a list assignment L''' that satisfies $|L'''(y_1)| = |L'''(y_2)| = 2$, $|L'''(y_3)| = 3$, $|L'''(w_1)| = |L'''(w_2)| = 2$ and $L'''(w_1) \neq L'''(w_2)$. And because of the same reason, $G'[X]$ is $L'|_X$ -colorable. If $c'_1 \notin L(w_1) = L(w_2)$, replacing c_1 by c'_1 , we can obtain that $G'[X]$ is $L'|_X$ -colorable similarly.

If c_2 or $c'_2 \notin L(w_1) = L(w_2)$, replacing A by $\{x_1, x_2\}$, and c_1 by c_2 or c'_2 , we can also obtain that $G'[X]$ is $L'|_X$ -colorable.

Finally, according to what we know about the list assignment L of G , let $L(x_1) = \{c_2, c'_2, c_{13}, c_{14}\}$, $L(x_2) = \{c_2, c'_2, c_{23}, c_{24}\}$, $L(x_3) = \{c_1, c'_1, c_{33}, c_{34}\}$, $L(x_4) = \{c_1, c'_1, c_{43}, c_{44}\}$, $L(w_1) = L(w_2) = \{c_1, c'_1, c_2, c'_2\}$. By Claim 3.1, $|L'(y_1) \cup L'(y_2) \cup L'(y_3)| \leq 6$. Thus, we know that $L'(y_2) \cap L'(y_3) \neq \emptyset$. Denote by d a color of $L'(y_2) \cap L'(y_3)$. Recall that $c_1 \notin L'(y_3)$, so $d \neq c_1$.

If $d \in \{c_2, c'_2, c'_1\}$, let $c(y_2) = c(y_3) = d$, $c(w_1) \in \{c_2, c'_2, c'_1\} - d$, $c(w_2) \in \{c_2, c'_2, c'_1\} - d - c(w_1)$, $c(y_1) \in L'(y_1) - c(w_1) - c(w_2)$, $c(x_1) \in \{c_{13}, c_{14}\} - c(y_1)$, $c(x_2) \in \{c_{23}, c_{24}\} - c(y_1)$. Then c is an $L'|_X$ -coloring of $G'[X]$.

If $d \notin \{c_2, c'_2, c'_1\}$ and $L'(y_1) \setminus \{c_2, c'_2, c'_1\} \neq \emptyset$. Let $c(y_2) = c(y_3) = d$, $c(x_1) = c(x_2) = c_2$, $c(w_1) = c'_2$, $c(w_2) = c'_1$, $c(y_1) \in L'(y_1) \setminus \{c_2, c'_2, c'_1\}$. Then c is an $L'|_X$ -coloring of $G'[X]$.

If $d \notin \{c_2, c'_2, c'_1\}$ and $L'(y_1) = \{c_2, c'_2, c'_1\}$. Let $c(y_2) = c(y_3) = d$, $c(w_1) = c_2$, $c(w_2) = c'_2$, $c(y_1) = c'_1$, $c(x_1) \in \{c_{13}, c_{14}\} - c'_1$, $c(x_2) \in \{c_{23}, c_{24}\} - c'_1$. Then c is an $L'|_X$ -coloring of $G'[X]$.

Combining all the discussions above, Theorem 3.1 holds. \square

Remark 3.1. In fact, if G has a list assignment as above, we can prove that G is L -colorable directly.

Theorem 3.2. If $k \geq 4$, $\text{ch}(K_{4,3,2*(k-4),1*2}) = k$.

Proof. For $G = K_{4,3,2*(k-4),1*2}$, denote its k parts as $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{y_1, y_2, y_3\}$, $U_i = \{u_i, v_i\}$ for $i = 1, 2, \dots, k-4$, $W_i = \{w_i\}$ for $i = 1, 2$. We will use induction on k . If $k = 4$, by Theorem 3.1, we are done. Suppose that $k \geq 5$, and Theorem 3.2 is true for smaller values of k . By contradiction, let $\text{ch}(K_{4,3,2*(k-4),1*2}) \neq k$, and L be a list assignment of G such that G is not L -colorable.

Claim 3.5. $L(u_i) \cap L(v_i) = \emptyset$ for every $i \in \{1, 2, \dots, k-4\}$.

Suppose that there exists a color $a \in L(u_i) \cap L(v_i)$. Then assign a to both u_i and v_i , and apply induction to $G - U_i$ and $L - a$. Thus we can obtain an L -coloring of G , a contradiction.

Let A be a largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \emptyset$, then we know that $2 \leq |A| \leq 3$ by Propositions 2.1 and 2.2. Choose a color $c_1 \in \bigcap_{x \in A} L(x)$ to color the vertices in A . Let $G' = G - A$, $L' = L - c_1$. As G is not L -colorable, G' is not L' -colorable. In particular, L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. In the following, we will prove that $G'[X]$ is $L'|_X$ -colorable. Then G' is L' -colorable by Lemma 2.1. Thus, we obtain a contradiction.

By the maximality of A , we have that $|L'(x)| = k$ for every $x \in V_1 \setminus A$. By Proposition 2.1, we assume, without loss of generality, that $|L'(y_1)| \geq k-1$, $|L'(y_2)| \geq k-1$ and $|L'(y_3)| = k$. And by Claim 3.5, we know that $|L'(u_i)| \geq k-1$, $|L'(v_i)| \geq k-1$ and $|L'(u_i) \cup L'(v_i)| \geq 2k-1$ for $i = 1, 2, \dots, k-4$. We also know that $|L'(w_i)| \geq k-1$ for $i = 1, 2$.

Claim 3.6. $|X \cap U_i| \leq 1$ for every $i \in \{1, 2, \dots, k-4\}$.

Otherwise, by Claim 3.5, $2k-1 \leq |L'(X)| < |X| \leq |V(G')| \leq 2k-1$. This is a contradiction.

Denote by t the size of $\bigcup_{1 \leq i \leq k-4} (X \cap U_i)$. Then it is clear that $0 \leq t \leq k-4$ by Claim 3.6. For every $z_i \in \bigcup_{1 \leq i \leq k-4} (X \cap U_i)$, choose a color b_i from $L'(z_i)$ and assign it to z_i such that b_1, b_2, \dots, b_t are pairwise different. Let $G'' = G' - (\bigcup_{1 \leq i \leq k-4} U_i)$, $X' = X \setminus (\bigcup_{1 \leq i \leq k-4} (X \cap U_i))$, $L'' = L' \setminus \{b_1, b_2, \dots, b_t\}$. In order to prove that $G'[X]$ is $L'|_X$ -colorable, we only need to show that $G''[X']$ is $L''|_{X'}$ -colorable.

As $G''[X']$ is a subgraph of G'' , it suffices to prove that G'' is L'' -colorable. Note that in $V(G'')$, $|L''(x)| \geq k - t \geq 4$ for every $x \in V_1 \setminus A$, $|L''(y_1)| \geq k - 1 - t \geq 3$, $|L''(y_2)| \geq k - 1 - t \geq 3$, $|L''(y_3)| \geq k - t \geq 4$, and $|L''(w_i)| \geq k - 1 - t \geq 3$ for $i = 1, 2$. In the proof of Theorem 3.1, replace G' by G'' and L' by L'' , we can show that G'' is L'' -colorable similarly. Thus, $G''[X]$ is $L''|_{X'}$ -colorable. \square

4. Ohba's conjecture is true for graphs $K_{5,3,2*(k-5),1*3}(k \geq 5)$

Similarly to the method in Section 3, in order to prove that $\text{ch}(K_{5,3,2*(k-5),1*3}) = k$ by induction, we shall show $\text{ch}(K_{5,3,1*3}) = 5$ first.

Theorem 4.1. $\text{ch}(K_{5,3,1*3}) = 5$.

Proof. For $G = K_{5,3,1*3}$, denote its five parts as $V_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $V_2 = \{y_1, y_2, y_3\}$, $W_i = \{w_i\}$ for $i = 1, 2, 3$. By contradiction, assume that L is a list assignment with $|L(u)| = 5$ for each $u \in V(G)$ such that G is not L -colorable.

Let A be a largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \emptyset$, then we know that $2 \leq |A| \leq 4$ by Propositions 2.1 and 2.2. Choose a color $c_1 \in \bigcap_{x \in A} L(x)$ to color the vertices in A . Let $G' = G - A$, $L' = L - c_1$. As G is not L -colorable, G' is not L' -colorable. In particular, L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. In the following, we will prove that $G'[X]$ is $L'|_X$ -colorable. Then G' is L' -colorable by Lemma 2.1. Thus we obtain a contradiction.

By the maximality of A , we have that $|L'(x)| = 5$ for every $x \in V_1 \setminus A$. And by Proposition 2.1, we assume, without loss of generality, that $|L'(y_1)| \geq 4$, $|L'(y_2)| \geq 4$ and $|L'(y_3)| = 5$. We also know that $|L'(w_i)| \geq 4$ for $i = 1, 2, 3$.

If $|X \cap V_2| \leq 2$, similarly to Cases 1 and 2 in the proof of Theorem 3.1, it is easy to show that $G'[X]$ is $L'|_X$ -colorable.

Suppose that $|X \cap V_2| = 3$. Namely, $\{y_1, y_2, y_3\} \subset X$.

Claim 4.1. $|X \cap V_1| \geq 2$ and $|A| \leq 3$.

Otherwise, $|X \cap V_1| \leq 1$ or $|A| = 4$, this implies that $|X| \leq 7$. If $L'(y_1) \cap L'(y_2) = \emptyset$, then $8 \leq |L'(X)| < |X| \leq 7$. This is a contradiction. If $L'(y_1) \cap L'(y_2) \neq \emptyset$, choose a color $b \in L'(y_1) \cap L'(y_2)$. Let $c(y_1) = c(y_2) = b$, $c(w_1) \in L'(w_1) - b$, $c(w_2) \in L'(w_2) - b - c(w_1)$, $c(w_3) \in L'(w_3) - b - c(w_1) - c(w_2)$. If there exists a vertex $x \in X \cap V_1$, let $c(x) \in L'(x) - b - c(w_1) - c(w_2) - c(w_3)$, $c(y_3) \in L'(y_3) - c(w_1) - c(w_2) - c(w_3) - c(x)$. If $X \cap V_1 = \emptyset$, let $c(y_3) \in L'(y_3) - c(w_1) - c(w_2) - c(w_3)$. Then c is an $L'|_X$ -coloring of $G'[X]$.

Claim 4.2. $L'(y_1) \cap L'(y_3) \neq \emptyset$ and $L'(y_2) \cap L'(y_3) \neq \emptyset$.

Otherwise, $9 \leq |L'(X)| < |X| \leq 9$. This is a contradiction.

By Claim 4.1, it suffices to consider the following three cases.

Case 1: $|A| = 3$ and $|X \cap V_1| = 2$.

$|A| = 3$ implies that $|X| \leq 8$. Without loss of generality, say $A = \{x_3, x_4, x_5\}$. By Claim 4.1, $X \cap V_1 = V_1 \setminus A = \{x_1, x_2\}$. Clearly, $|L'(x_1) \cap L'(x_2)| \geq 3$, and $L'(y_1) \cap L'(y_2) \neq \emptyset$. Otherwise, $8 \leq |L'(X)| < |X| \leq 8$. This is a contradiction.

Subcase 1.1: $L'(w_1)$, $L'(w_2)$ and $L'(w_3)$ are not the same color lists.

Choose a color $c_2 \in L'(y_1) \cap L'(y_2)$ and a color $c_3 \in L'(x_1) \cap L'(x_2)$ such that $c_3 \neq c_2$. Assign c_2 to both y_1 and y_2 , and c_3 to both x_1 and x_2 . Since $L'(w_1)$, $L'(w_2)$ and $L'(w_3)$ are not the same color lists, we know that $|L'(w_i) \setminus \{c_2, c_3\}| \geq 2$ for $i = 1, 2, 3$, and $L'(w_1) \setminus \{c_2, c_3\}$, $L'(w_2) \setminus \{c_2, c_3\}$, $L'(w_3) \setminus \{c_2, c_3\}$ are not the same color lists. Hence, for $i = 1, 2, 3$, we can choose a color d_i from $L'(w_i) \setminus \{c_2, c_3\}$ to color w_i . And we can choose a color d_4 from $L'(y_3) \setminus \{c_3, d_1, d_2, d_3\}$ to color y_3 afterwards. Thus, $G'[X]$ is $L'|_X$ -colorable.

Subcase 1.2: $L'(w_1) = L'(w_2) = L'(w_3)$.

Let $\{1, 2, 3, 4\} \subseteq L'(w_i)$ for $i = 1, 2, 3$. Clearly, $(L'(y_1) \cap L'(y_2)) \subseteq \{1, 2, 3, 4\}$ and $(L'(x_1) \cap L'(x_2)) \subseteq \{1, 2, 3, 4\}$. Otherwise, it is easy to see that $G'[X]$ is $L'|_X$ -colorable similarly to Subcase 1.1.

According to the size of $L'(x_1) \cap L'(x_2)$, we need consider two subcases as follows:

Subcase 1.2.1: $|L'(x_1) \cap L'(x_2)| = 3$.

Without loss of generality, denote $L'(x_1) = \{1, 2, 3, c_{14}, c_{15}\}$, $L'(x_2) = \{1, 2, 3, c_{24}, c_{25}\}$. Choose a color $c_2 \in (L'(y_1) \cap L'(y_2)) \subseteq \{1, 2, 3, 4\}$. Let $c(y_1) = c(y_2) = c_2$, $c(w_1) \in \{1, 2, 3, 4\} - c_2$, $c(w_2) \in \{1, 2, 3, 4\} - c_2 - c(w_1)$,

$c(w_3) \in \{1, 2, 3, 4\} - c_2 - c(w_1) - c(w_2)$. As $c_2 \notin L'(y_3)$ by Proposition 2.1, we have $|L'(y_3) \setminus \{1, 2, 3, 4\}| \geq 2$. Let $\{b_{34}, b_{35}\} \subseteq (L'(y_3) \setminus \{1, 2, 3, 4\})$. Note that $c_{14}, c_{15}, c_{24}, c_{25}$ are pairwise different and $\{c_{14}, c_{15}, c_{24}, c_{25}\} \cap \{1, 2, 3, 4\} \leq 1$. Without loss of generality, say $\{c_{15}, c_{24}, c_{25}\} \cap \{1, 2, 3, 4\} = \emptyset$. So we can let $c(x_1) = c_{15}$, $c(y_3) \in \{b_{34}, b_{35}\} - c(x_1)$, $c(x_2) \in \{c_{24}, c_{25}\} - c(y_3)$. Thus, c is an $L'|_X$ -coloring of $G'[X]$.

Subcase 1.2.2: $|L'(x_1) \cap L'(x_2)| \geq 4$.

Let $L'(x_1) = \{1, 2, 3, 4, c_{15}\}$, $L'(x_2) = \{1, 2, 3, 4, c_{25}\}$.

If $|L'(y_1) \cap L'(y_2)| \geq 2$, then $|L'(y_3) \setminus \{1, 2, 3, 4\}| \geq 3$ by Proposition 2.1. Denote $\{b_{33}, b_{34}, b_{35}\} \subseteq (L'(y_3) \setminus \{1, 2, 3, 4\})$. Use colors in $\{1, 2, 3, 4\}$ to color y_1, y_2, w_1, w_2 and w_3 . Color x_1, x_2 and y_3 with c_{15}, c_{25} and a color in $\{b_{33}, b_{34}, b_{35}\} \setminus \{c_{15}, c_{25}\}$, respectively. Thus, $G'[X]$ is $L'|_X$ -colorable.

If $|L'(y_1) \cap L'(y_2)| = 1$, without loss of generality, let $\{1, b_{13}, b_{14}, b_{15}\} \subseteq L'(y_1)$, $\{1, b_{23}, b_{24}, b_{25}\} \subseteq L'(y_2)$, where $b_{13}, b_{14}, b_{15}, b_{23}, b_{24}$ and b_{25} are pairwise different. Clearly, $|\{b_{13}, b_{14}, b_{15}, b_{23}, b_{24}, b_{25}\} \setminus \{1, 2, 3, 4\}| \geq 3$. If $L'(y_1) \setminus \{1, 2, 3, 4\} \neq \emptyset$ and $L'(y_2) \setminus \{1, 2, 3, 4\} \neq \emptyset$, say $b_{15} \in L'(y_1) \setminus \{1, 2, 3, 4\}$ and $b_{25} \in L'(y_2) \setminus \{1, 2, 3, 4\}$, use colors in $\{1, 2, 3, 4\}$ to color x_1, x_2, w_1, w_2 and w_3 . Color y_1, y_2 and y_3 with b_{15}, b_{25} and a color in $L'(y_3) \setminus \{1, 2, 3, 4\}$, respectively. Thus, we obtain that $G'[X]$ is $L'|_X$ -colorable. If $L'(y_1) \setminus \{1, 2, 3, 4\} = \emptyset$ or $L'(y_2) \setminus \{1, 2, 3, 4\} = \emptyset$, say $L'(y_1) \setminus \{1, 2, 3, 4\} = \emptyset$, namely, $L'(y_1) = \{1, 2, 3, 4\}$. This implies that $\{b_{23}, b_{24}, b_{25}\} \cap \{1, 2, 3, 4\} = \emptyset$. By Claim 4.2, $L'(y_1) \cap L'(y_3) \neq \emptyset$, so $L'(y_1) \cap L'(y_3) \subseteq \{2, 3, 4\}$. Use colors in $\{1, 2, 3, 4\}$ to color y_1, y_3, w_1, w_2 and w_3 . Color x_1, x_2 and y_2 with c_{15}, c_{25} and a color in $\{b_{23}, b_{24}, b_{25}\} \setminus \{c_{15}, c_{25}\}$, respectively. We also obtain that $G'[X]$ is $L'|_X$ -colorable.

Case 2: $|A| = 2$ and $|X \cap V_1| = 2$.

$|A| = 2$ and $|X \cap V_1| = 2$ also implies that $|X| \leq 8$. Without loss of generality, say $A = \{x_4, x_5\}$, $X \cap V_1 = \{x_1, x_2\}$. Similarly to Case 1 completely, it is easy to see that $G'[X]$ is $L'|_X$ -colorable.

Case 3: $|A| = 2$ and $|X \cap V_1| = 3$.

In this case, without loss of generality, say $A = \{x_4, x_5\}$, $X \cap V_1 = V_1 \setminus A = \{x_1, x_2, x_3\}$. As $|X| \leq |V(G')| = 9$, we have that $|L'(x_i) \cap L'(x_j)| \geq 2$ for every $i, j = 1, 2, 3, i \neq j$. Otherwise, $9 \leq |L'(X)| < |X| \leq 9$. This is a contradiction. Denote $L(x_i) = \{c_{i1}, c_{i2}, c_{i3}, c_{i4}, c_{i5}\}$ for $i = 1, 2, 3, 4, 5$. Note that $L(x_1) \cap L(x_2) \cap L(x_3) = \emptyset$ as $|A| = 2$ and A is largest. Since $|L'(x_i) \cap L'(x_j)| \geq 2$ for every $i, j = 1, 2, 3, i \neq j$, without loss of generality, let $c_{11} = c_{21} = 1, c_{12} = c_{22} = 2, c_{13} = c_{31} = 3, c_{14} = c_{32} = 4, c_{23} = c_{33} = 5, c_{24} = c_{34} = 6$.

As A is a largest subset of V_1 and $|A| = 2$, we can also choose $A = \{x_1, x_2\}$ and $c_1 = 1$ in the beginning. In the condition of G being not L -colorable, with the same method, we obtain either a previously considered case or that $|A| = 2$ and $|X \cap V_1| = 3$, where X is a maximal subset of $V(G') = V(G - A)$ such that $|L'(X)| < |X|$. Hence, $|L'(x_i) \cap L'(x_j)| \geq 2$ for every $i, j = 3, 4, 5, i \neq j$. Consider $L(x_3)$ and $L(x_4)$. Since $|L'(x_3) \cap L'(x_4)| \geq 2$, let $\{a, b\} \subseteq L'(x_3) \cap L'(x_4)$. Since A is maximal, we have $\{a, b\} \cap \{1, 2, 3, 4, 5, 6\} = \emptyset$. Thus, $\{3, 4, 5, 6, a, b\} \subseteq L(x_3)$, and hence $|L(x_3)| \geq 6$. This is a contradiction.

Combining all the discussions above, Theorem 4.1 holds. \square

Theorem 4.2. If $k \geq 5$, $\text{ch}(K_{5,3,2*(k-5),1*3}) = k$.

Proof. The proof of Theorem 4.2 is similar to the proof of Theorem 3.2 completely, so we omit it. \square

5. Some discussions about Theorem 1.5

Firstly, we point out that Theorem 1.5 [2] is also true for $m \leq 2s + 2$. Namely, Theorem 1.5 can be improved as follows.

Theorem 5.1. If $m \leq 2s + 2$, $\text{ch}(K_{m,2*(k-s-1),1*s}) = k$.

Here we need not give a new proof for Theorem 5.1. The fact of the matter is that, in the procedure of proving Theorem 1.5 itself, its Case 1 is true if we replace $m \leq 2s + 1$ by $m \leq 2s + 3$, and its Case 2 is true if replace $m \leq 2s + 1$ by $m \leq 2s + 2$. Thus, Theorem 1.5 is true for $m \leq 2s + 2$ as a whole.

Secondly, we show that, in Theorem 5.1, if $m \leq s + 3$, namely, if restricting $|V(G)|$ within $2\chi(G) + 1$, the proof is very easy and short. In other words, for $K_{s+3,2*(k-s-1),1*s}$, to verify Ohba's conjecture is very easy.

Theorem 5.2. $\text{ch}(K_{s+3,2*(k-s-1),1*s}) = k$.

Proof. For $G = K_{s+3,2*r,1*s}$, where $r = k - s - 1$. Denote its k parts as $V_1 = \{x_1, x_2, \dots, x_{s+3}\}$, $U_i = \{u_i, v_i\}$ for $i = 1, 2, \dots, r$, $W_i = \{w_i\}$ for $i = 1, 2, \dots, s$. If $r = 0$, it is clear that $\text{ch}(K_{s+3,1*s}) = s + 1 = k$. Suppose that $r \geq 1$ and Theorem 5.2 is true for smaller values of r . By contradiction, suppose that $\text{ch}(K_{s+3,2*(k-s-1),1*s}) \neq k$, and L is a list assignment of G such that G is not L -colorable.

Claim 5.1. $L(u_i) \cap L(v_i) = \emptyset$ for every $i = 1, 2, \dots, r$.

Otherwise, it is easy to see that G is L -colorable by induction.

Claim 5.2. $|A| \geq 2$ where A is a largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \emptyset$.

Otherwise, $L(x_1), L(x_2), \dots, L(x_{s+3})$ are pairwise disjoint. Clearly, L satisfies Hall's condition and hence G is L -colorable.

Let A be a largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \emptyset$. By Claim 5.2, $|A| \geq 2$. Choose a color $c \in \bigcap_{x \in A} L(x)$, and let $G' = G - A$, $L' = L - c$. Since G is not L -colorable, G' is not L' -colorable. In particular, L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. Clearly, $|X \cap U_i| \leq 1$ for $i = 1, 2, \dots, r$. Otherwise $2k - 1 \leq |L'(X)| < |X| \leq |V(G')| \leq 2k - 1$. This is a contradiction. Hence $|X \setminus V_1| \leq k - 1$. Note that $|L'(u)| \geq k - 1$ for any $u \in X \setminus V_1$ and $|L'(u)| = k$ for every $u \in X \cap V_1$. It is obvious that $G'[X]$ is $L'|_X$ -colorable. By Lemma 2.1, G' is L' -colorable. This is a contradiction. \square

Acknowledgments

We would like to thank the referees for their careful reading and valuable comments. In particular, we are indebted to one of them for pointing out a flaw in the proof of Theorem 3.2.

References

- [1] N. Alon, Restricted colorings of graphs, in: K. Waller (Ed.), *Surveys in Combinatorics*, London Mathematical Society Lecture Note Series, vol. 187, Cambridge University Press, Cambridge, UK, 1993, pp. 1–33.
- [2] H. Enotomo, K. Ohba, K. Ota, J. Sakamoto, Choice number of some complete multipartite graphs, *Discrete Math.* 244 (2002) 55–56.
- [3] P. Erdős, A.L. Rubin, H. Taylor, Choosability in graphs, *Congr. Numer.* 26 (1979) 125–157.
- [4] F. Galvin, The list chromatic index of a bipartite multigraph, *J. Combin. Theory Ser. B* 63 (1995) 153–158.
- [5] S. Gravier, F. Maffray, Graphs whose choice number is equal to their chromatic number, *J. Graph Theory* 27 (1998) 87–97.
- [6] R. Häggkvist, A. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs, *J. Graph Theory* 16 (1992) 503–516.
- [7] H.A. Kierstead, On the choosability of complete multipartite graphs with part size three, *Discrete Math.* 211 (2000) 255–259.
- [8] K. Ohba, On chromatic-choosable graphs, *J. Graph Theory* 40 (2002) 130–135.
- [9] K. Ohba, Choice number of complete multipartite graphs with part size at most three, *Ars Combin.* 72 (2004) 133–139.
- [10] B. Reed, B. Sudakov, List colouring when the chromatic number is close to the order of the graph, *Combinatorica* 25 (2005) 117–123.
- [11] V.G. Vizing, Coloring the vertices of a graph in prescribed colors, *Diskret. Anal.* 29 (1976) 3–10 (in Russian).
- [12] D.R. Woodall, List colourings of graphs, in: J.W.P. Hirschfeld (Ed.), *Surveys in Combinatorics*, London Mathematical Society Lecture Note Series, vol. 288, Cambridge University Press, Cambridge, UK, 2001, pp. 269–301.